PBW deformations of $\mathcal{U}(\mathfrak{g})$ smash products in OSCAR

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ARTIG #4

Chair for Algebra and Representation Theory
RWTH Aachen University

based on ongoing joint work with J. Flake

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Structure of the talk

- Setup and problem description
  - PBW deformations
  - Smash products
  - Infinitesimal Hecke algebras
- Parameterizing Hom spaces: Interpolation techniques
- Implementation

PBW deformations

Definition (PBW deformation)

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Let $\mathfrak{g}$ be a Lie algebra. Then its universal enveloping algebra $A' = \mathcal{U}(\mathfrak{g})$ is a PBW deformation of $A = S(\mathfrak{g})$.

PBW deformations (2)

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Let

$$R = \operatorname{span}\{v \otimes w - w \otimes v\} \cong \mathfrak{g} \wedge \mathfrak{g} \quad \subset \quad \mathfrak{g} \otimes \mathfrak{g}.$$

We can now write

$$A' = \mathcal{U}(\mathfrak{g}) \cong T(\mathfrak{g}) \Big/ \langle r - \kappa(r) \mid r \in R \rangle ,$$

$$A = S(\mathfrak{g}) \cong T(\mathfrak{g}) \Big/ \langle R \rangle ,$$

where $\kappa : \mathfrak{g} \wedge \mathfrak{g} \to \mathfrak{g}$ is the Lie bracket.

Smash products

Definition (H-module algebra)

Let $(H, m, \Delta, u, \epsilon, S)$ be a Hopf algebra with a bijective antipode S. We say that a left H-module A is a **left H-module algebra** if

$$h \cdot 1_A = \epsilon(h) 1_A \qquad \text{and} \qquad h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$$

for all $h \in H$, $a, b \in A$ where $\Delta(h) = \sum h_1 \otimes h_2$ (in Sweedler's notation).

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Definition (Smash product)

For some H-module algebra A, the **smash product** $A \# H$ is the tensor product $A \otimes H$ with multiplication given by

$$(a \otimes h)(a' \otimes h') := a(h_1 \cdot a') \otimes h_2 h'$$

for all $a, a' \in A$, $h, h' \in H$.

Deformations of smash products

Let V be an H-module, $\dim_k V < \infty$, $R \subset V \otimes V$ a subspace preserved by H.

$B = T(V)/\langle R \rangle$ is an $\mathbb{N}$-graded, Koszul, left H-module algebra with $B^{(0)} = k$ and the action of H preserves the grading.

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Take $\kappa : R \to H$ a k-bilinear map. Define

$$A_\kappa := T(V)\#H \Big/ \langle r - \kappa(r) \mid r \in R \rangle$$

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$\rightsquigarrow$ We now consider A_κ as a filtered deformation of A_0.

Infinitesimal Hecke algebras

From now on: $\mathfrak{g}$ a Lie (super)algebra, V a rep. of $\mathfrak{g}$, $H := \mathcal{U}(\mathfrak{g})$, $R = V \wedge V \subset V \otimes V$.

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$$A_\kappa = T(V) \rtimes U(\mathfrak{g}) \Big/ \langle r - \kappa(r) \mid r \in R \rangle \, ,$$
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Definition

A PBW deformation A_κ of A_0 in this setup is called an **infinitesimal Hecke algebra**.

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For a given pair $(\mathfrak{g}, V)$, classify all infinitesimal Hecke algebras.

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[1] Pavel Etingof, Wee Liang Gan, and Victor Ginzburg. Continuous Hecke algebras. In: *Transform. Groups* 10.3-4 (2005), pp. 423–447

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- interpolating PBW deformations for $(\mathfrak{osp}_{m,2n}, \bigwedge^e k^{m+2n})$[3]

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[3] Johannes Flake and Verity Mackscheidt. *Interpolating PBW Deformations for the Orthosymplectic Groups*. 2022. arXiv: 2206.08226 [math.RT]

Main ingredient for computations

Theorem (Walton-Witherspoon)

A_κ is a PBW deformation of A_0 if and only if

a) *κ is H-invariant, i.e., $h \cdot \kappa(r) = \kappa(h \cdot r)$ for all $r \in R$ and $h \in H$, and*

c') *$(\kappa^C \otimes \mathrm{id} - \mathrm{id} \otimes \kappa^C) = 0$ on $(R \otimes V) \cap (V \otimes R)$.*

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Remaining problem: find "efficient" generating sets of $\mathrm{Hom}(V \wedge V, \mathcal{U}_{\leq d}(\mathfrak{g}))$ for increasing d.

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For $\mathfrak{g} = \mathfrak{gl}_n$: take

$$\mathrm{Hom}(V \wedge V, S^d(\mathfrak{g})) \cong \mathrm{Hom}(V \wedge V, S^d(V_{\mathrm{nat}} \otimes V_{\mathrm{nat}}^*))$$

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Idea[4]: interpolate $\mathrm{Rep}(GL_n)$ to non-integer values of n

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Tensor structure given by horizontal glueing

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Theorem (Deligne-Milne)

If $t = n \in \mathbb{N}_0$, then there is a full and essentially surjective functor

$$\underline{\mathrm{Rep}}(GL_{t=n}) \to \mathrm{Rep}(GL_n).$$

Specialising an arc diagram

$$F : \langle \text{arc diagram in } \operatorname{Hom}(r, s) \rangle_k \twoheadrightarrow \operatorname{Hom}_{GL_n}(V^{\otimes r}, V^{\otimes s})$$

where

$$V := \text{natural representation}$$
$$V^{\otimes s} := \bigotimes_{i=1,\dots,|s|} (1 - s_i) \cdot V \oplus s_i \cdot V^*$$

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Fix a basis $e_1,\dots,e_n$ of V and let $e_1^*,\dots,e_n^*$ be the dual basis.
$\rightsquigarrow$ basis $\{e_{(i)}\}$ of $V^{\otimes s}$ where i is a $|s|$-string in $\{1,\dots,n\}$

Specialising an arc diagram (2)

Definition

Arc diagram π is **compatible** with $|r|$-string i and $|s|$-string j if connected points in π have the same label w.r.t. i and j.

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$$F(\pi) = \sum_{\substack{|r|\text{-string } i, \\ |s|\text{-string } j, \\ \pi \text{ compatible with } i,j}} \left(e_{(k)} \mapsto \delta_{i,k} e_{(j)} \right)$$

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- $F(\cup \in \mathrm{Hom}(01,)) = (e_i \otimes e_j^* \mapsto \delta_{i,j})$

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- $F(\cup \in \mathrm{Hom}(01,)) = (e_i \otimes e_j^* \mapsto \delta_{i,j})$
- $F(\cap \in \mathrm{Hom}(,01)) = (1 \mapsto \sum_{i=1,\dots,n} e_i \otimes e_i^*)$

Symmetric and exterior powers

Consider

$$\mathrm{Hom}_{GL_n}((V \otimes V^*) \wedge (V \otimes V^*), V \otimes V^*) \subset \mathrm{Hom}_{GL_n}((V \otimes V^*) \otimes (V \otimes V^*), V \otimes V^*).$$

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Example

Summary

If V is build from V_{nat} using $\otimes, \wedge, S, *, (\oplus)$, every element from

$$\mathrm{Hom}(V \wedge V, S^d(V_{\mathrm{nat}} \otimes V_{\mathrm{nat}}^*))$$

can be represented as a linear combination of orbits of arc diagrams.

Summary

If V is build from V_{nat} using $\otimes, \wedge, S, *, (\oplus)$, every element from

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can be represented as a linear combination of orbits of arc diagrams.

In particular, this is the case for any PBW deformation map κ.

PBWDeformations.jl

All of following is implemented in the julia package PBWDeformations.jl[5] built upon OSCAR[6].

```
julia> using Pkg; Pkg.add(["Oscar", "PBWDeformations"]) # installation
... [wait some time] ...

julia> using Oscar, PBWDeformations
  ___   ____   ____    _    ____
 / _ \ / ___| / ___|  / \  |  _ \   |  Combining ANTIC, GAP, Polymake, Singu
| | | |\___ \| |     / _ \ | |_) |  |  Type "?Oscar" for more information
| |_| | ___) | |___ / ___ \|  _ <   |  Manual: https://docs.oscar-system.org
 \___/ |____/ \____/_/   \_\_| \_\  |  Version 1.0.3

julia>
```

[5] https://github.com/lgoettgens/PBWDeformations.jl
[6] https://www.oscar-system.org/

Construct a smash product

$$\mathtt{g} := \mathfrak{gl}_3(\mathbb{Q}), \quad \mathtt{V} := V_{\mathrm{nat}} \oplus V_{\mathrm{nat}}^*, \quad \mathtt{sp} := T(\mathtt{V}) \rtimes \mathcal{U}(\mathtt{g})$$

```
julia> g = general_linear_lie_algebra(QQ, 3)
General linear Lie algebra of degree 3
  of dimension 9
over rational field

julia> V = direct_sum(standard_module(g), dual(standard_module(g)))
Direct sum module
  of dimension 6
  direct sum with direct summands
    standard module
    dual of
      standard module
over general linear Lie algebra of degree 3 over QQ

julia> sp = smash_product(g, V)
Smash Product of general linear Lie algebra of degree 3 over QQ and
direct sum module of dimension 6 over gl_3
```

Arithmetics in a smash product

- do all computations in a free associative algebra
- lazily simplify to expression in ordered monomials
- using rewriting rules based on commutator relations

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```
julia> gens(sp) |> println
SmashProductLieElem{...}[x_1_1, x_1_2, x_1_3, x_2_1, x_2_2, x_2_3,
 x_3_1, x_3_2, x_3_3, (v_1^(1)), (v_2^(1)), (v_3^(1)), ((v_1*)^(2)),
 ((v_2*)^(2)), ((v_3*)^(2))]

julia> some_elem = gen(sp,4) * gen(sp,10) - gen(sp,10) * gen(sp,4)
x_2_1*(v_1^(1)) - (v_1^(1))*x_2_1

julia> simplify(gen(sp,4) * gen(sp,10))
x_2_1*(v_1^(1))

julia> simplify(gen(sp,10) * gen(sp,4))
x_2_1*(v_1^(1)) - (v_2^(1))

julia> simplify(some_elem)
(v_2^(1))
```

Specializing arc diagrams

Specialize the orbit of `"aB,AdDb"` to $\mathrm{Hom}(V_{\mathrm{nat}} \otimes V_{\mathrm{nat}}^*, S^2\mathfrak{g})$

```
julia> PBWDeformations.arcdiag_to_deformationmap(PBWDeformations.GL(),
         arc_diagram(Directed, "aB,AdDb"), sp,
         tensor_product(standard_module(g), dual(standard_module(g))))
[x_1_1^2+x_1_2*x_2_1+x_1_3*x_3_1-x_1_1+1//2*x_2_2+1//2*x_3_3  ...  ...]
[             x_1_1*x_2_1+x_2_1*x_2_2+x_2_3*x_3_1+1//2*x_2_1  ...  ...]
[             x_1_1*x_3_1+x_2_1*x_3_2+x_3_1*x_3_3+3//2*x_3_1  ...  ...]
```

Let `A` be the 3×3 output matrix. Define the resulting hom on the standard basis via

$$e_i \otimes e_j^* \mapsto \mathtt{A[i,j]}$$

and extend linearly.

Putting everything together

Recall: $\mathtt{g} := \mathfrak{gl}_3(\mathbb{Q})$, $\mathtt{V} := V_{\mathrm{nat}} \oplus V^*_{\mathrm{nat}}$, $\mathtt{sp} := T(\mathtt{V}) \rtimes \mathcal{U}(\mathtt{g})$

[7] Pavel Etingof, Wee Liang Gan, and Victor Ginzburg. Continuous Hecke algebras. In: *Transform. Groups* 10.3-4 (2005), pp. 423–447.

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1-element Vector{MatElem{<:FreeAssAlgElem{QQFieldElem}}}:
[0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; -1 0 0 0 0 0; 0 -1 0 0 0 0; 0 0 -1 
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[0 0 0 2*x_1_1+x_2_2+x_3_3 x_1_2 x_1_3; 0 0 0 x_2_1 x_1_1+2*x_2_2+x_3_3 x_2_
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$\rightsquigarrow$ reproduces the results from Etingof-Gan-Ginzburg[7]

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- Write some proper documentation

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Questions?